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# Some Aspects of Wiener-Hopf Factorization

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# Some aspects of Wiener–Hopf factorization

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Wiener–Hopf factorization means many apparently different things, both in theory and in its wide variety of applications. This paper is designed so that almost all of it may be read by non-probabilists, though it makes demands on the reader's ability to use analogy. It is written in response to requests from people in other fields to give some idea of what probabilists are doing. It gives some reformulations of the probabilistic Wiener–Hopf problem studied by London *et al*. One reformulation as a problem of simultaneous reduction of quadratic forms is used to motivate another as a Riemann–Hilbert problem. In addition to trying to synthesize various results, it answers affirmatively a question of McGregor as to whether a useful convolution formula which he obtained in a special case holds generally. Section 4 on examples, methods, and their interrelations is the liveliest part of the paper. Though algebra and complex analysis are successful and link perfectly with probability in much of what has so far been achieved, the scope of these methods is very severely limited, and much more challenging problems lie ahead. Motivation for this study derives originally from the practically important fact that integrals of Markov processes often provide better models than Markov processes themselves; but it has obvious pure-mathematical 'rightness' too.

## 1. An algebraic result

Let $E$ be a finite set, and let $E = E^+ \cup E^-$ be a disjoint partition of $E$ such that neither $E^-$ nor $E^+$ is empty. Let $V: E \to \mathbb{R}$ be such that $V > 0$ on $E^+$ and $V < 0$ on $E^-$. We shall also use $V$ to denote the diagonal $E \times E$ matrix $\mathrm{diag}(V_i : i \in E)$ which represents multiplication by the function $V$. Let $\mathscr{V}$ be the vector space of real functions (vectors) on $E$ and let $\mathscr{V}_\pm$ be the corresponding space for $E^\pm$. Let $m$ be a measure on $E$ with $m_i = m(\{i\}) > 0$ for every $i$ in $E$. Introduce the standard inner product

$$\langle f, g \rangle := \sum_{i \in E} f_i g_i m_i$$

on $\mathscr{V}$. (We write ':=' for 'is defined to equal'.)

Let $Q$ be an $E \times E$ matrix which is a strict sub-$Q$-matrix on $E$ in that

$$q_{ij} \geqslant 0 \quad (i \neq j), \quad \delta_i := \sum_{j \in E} q_{ij} < 0 \quad (\forall i).$$

Suppose further that $Q$ is $m$-symmetrizable in that

$$m_i q_{ij} = m_j q_{ji} \quad \forall (i, j),$$

so that $Q$ is self-adjoint for $\langle \cdot, \cdot \rangle$. The *Dirichlet form* ('energy integral')

$$\mathscr{E}(f, g) := -\langle f, Qg \rangle \tag{1.1}$$

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defines a proper inner-product on $\mathscr{V}$ since

$$\mathscr{E}(f,f) = \tfrac{1}{2}\sum_{i\neq j}\sum m_i q_{ij}(f_j - f_i)^2 - \sum_i m_i \delta_i f_i^2.$$

If we allowed each $\delta_i$ to equal 0, and we shall later do this for the 'continuous' analogue, then $\mathscr{E}$ would be non-negative definite with $\mathscr{E}(1,1) = 0$ for the constant function 1.

**Theorem.** 1. *There is one and only one way of finding* (i) *an isomorphism*

$$f \leftrightarrow (f_+, f_-) \quad \text{from} \quad \mathscr{V} \text{ to } \mathscr{V}_+ \oplus \mathscr{V}_-$$

*such if $f_\mp = 0$ then $f_\pm = f^\pm$, where $f^\pm$ denotes the restriction of $f$ to $E^\pm$, and* (ii) *proper inner products $\langle\cdot,\cdot\rangle_\pm$ and $\mathscr{E}_\pm$ on $\mathscr{V}_\pm$, such that*

$$\langle f, Vg\rangle = \langle f_+, g_+\rangle_+ - \langle f_-, g_-\rangle_-, \tag{1.2}$$

*and*

$$\mathscr{E}(f,g) = \mathscr{E}_+(f_+, g_+) + \mathscr{E}_-(f_-, g_-). \tag{1.3}$$

2. *There exist strictly substochastic* (*componentwise non-negative, with row sums less than* 1) *$E^\mp \times E^\pm$ matrices $\Pi^\pm$ such that, if $I^\pm$ denotes the identity $E^\pm \times E^\pm$ matrix, then*

$$f = \begin{pmatrix} I^+ & \Pi^- \\ \Pi^+ & I^- \end{pmatrix}\begin{pmatrix} f_+ \\ f_- \end{pmatrix}, \quad f \in \mathscr{V}. \tag{1.4}$$

*Moreover, $\Pi^+$ and $\Pi^-$ are $|V|m$ adjoint: for $i \in E^-$ and $j \in E^+$,*

$$|V_i| m_i \Pi^+_{ij} = |V_j| m_j \Pi^-_{ji}. \tag{1.5}$$

3. *There exist strict sub-Q-matrices $G^\pm$ on $E^\pm$, self-adjoint relative to $\langle\cdot,\cdot\rangle_\pm$, such that*

$$\mathscr{E}_\pm(f^\pm, g^\pm) = -\langle f^\pm, G^\pm g^\pm\rangle_\pm \tag{1.6}$$

*whenever $f^\pm, g^\pm \in \mathscr{V}_\pm$, and that*

$$(V^{-1}Qf)_\pm = \pm G^\pm f_\pm, \quad f \in \mathscr{V}. \tag{1.7}$$

4. *The following Wiener–Hopf factorization of $V^{-1}Q$ holds:*

$$\begin{pmatrix} I^+ & \Pi^- \\ \Pi^+ & I^- \end{pmatrix}^{-1} V^{-1}Q \begin{pmatrix} I^+ & \Pi^- \\ \Pi^+ & I^- \end{pmatrix} = \begin{pmatrix} G^+ & 0 \\ 0 & -G^- \end{pmatrix}. \tag{1.8}$$

Results (1.2) and (1.3) exhibit the problem as one with which we are very familiar from early courses in applied mathematics, that of simultaneous reduction of two quadratic forms, one of which is positive-definite; but, of course, we have much additional structure here.

Though (1.6) is in one sense a 'Dirichlet form' expression analogous to (1.1), the inner product on the right-hand side is of course not a standard $L^2$ product. Indeed, for $f^+, g^+ \in \mathscr{V}_+$, we have

$$\langle f^+, g^+\rangle_+ = \langle f^+, (I^+ - \Pi^-\Pi^+)g^+\rangle_{|Vm|^+}, \tag{1.9}$$

where

$$\langle f^+, h^+\rangle_{|Vm|^+} = \sum_{j\in E^+} f_j h_j |V_j| m_j.$$

For probabilistic interpretation of the fact that $G^+$ is self-adjoint relative to the $\langle\cdot,\cdot\rangle_+$ inner product, see forthcoming work by Joanne Kennedy. In terms of algebra, the $\langle \cdot, \cdot \rangle_+$ self-adjointness of $G^+$ is equivalent to (1.5), and (1.5) is algebraically trivial. But in more general contexts within probability, the analogue of result (1.5) is intuitively obvious but difficult to prove.

## 2. Proof of Theorem 1

### (a) *A uniqueness result*

Suppose that an isomorphism as at Part 1 (i) of the theorem and inner products as at Part 1 (ii) exist so that (1.2) and (1.3) hold. Let $G^+$ be the negative-definite self-adjoint (relative to $\langle \cdot, \cdot \rangle_+$) $E^+ \times E^+$ matrix such that (1.6) holds.

Suppose that $\lambda$ is an eigenvalue (necessarily real) of $G^+$, so that $\lambda < 0$; and let $h^+$ in $\mathscr{V}_+$ denote a corresponding (real) eigenvector. Let $g$ correspond to $(h^+, 0)$ under our isomorphism, so that

$$g_+ = g^+ = h^+, \quad g_- = 0.$$

Then, for any $f$ in $\mathscr{V}$

$$\begin{aligned}\langle f, \lambda V g\rangle &= \langle f_+, \lambda g_+\rangle_+ = \langle f_+, G^+ g_+\rangle_+ \\ &= -\mathscr{E}_+\langle f_+, g_+\rangle = -\mathscr{E}(f,g) = \langle f, Qg\rangle,\end{aligned}$$

whence $\lambda V g = Qg$, so that $\lambda$ is an eigenvalue of $V^{-1}Q$ with corresponding eigenvector $g$. From what has just been proved and the corresponding 'minus' result, it follows that $V^{-1}Q$ has enough eigenvectors to span $\mathscr{V}$. It also follows that if $g$ in $\mathscr{V}$ is an eigenvector of $V^{-1}Q$ corresponding to a real eigenvalue $\lambda < 0$, then $g^+ = g_+$ is an eigenvector for $G^+$ with eigenvalue $\lambda$, $g_- = 0$, and for any $f$ in $\mathscr{V}$

$$\lambda \langle f_+, g_+\rangle_+ = -\mathscr{E}(f,g).$$

The uniqueness assertion of Part 1 of the theorem now follows.

We now turn to the 'existence' proof, and though we have to start again from scratch, we can watch the structure just discovered emerge from the probability.

### (b) *Use of probability*

Non-probabilists need only browse in this subsection and the next. Indeed, they can just jump to §2$d$ below.

Let $X$ be a Markov chain on $E$ with $Q$-matrix $Q$. Let $\mathbb{P}^i$ denote the probability law of $X$ when $X$ starts at $i$. Define

$$\varphi_t := \int_0^t V(X_s)\,\mathrm{d}s, \quad \tau_t^\pm := \inf\{u : \pm\varphi_u > t\}, \tag{2.1}$$

and for $i \in E^-$ and $j \in E^+$, define the 'half-winding probabilities':

$$\Pi^+_{ij} := \mathbb{P}^i\{X(\tau_0^+) = j\}, \quad \Pi^-_{ji} := \mathbb{P}^j\{X(\tau_0^-) = i\}. \tag{2.2}$$

The 'half-winding' terminology is natural if one studies the phase picture $(\varphi, X)$ (see McKean 1963). The process $\{X(\tau_t^\pm) : t \geqslant 0\}$ is a Markov chain on $E^\pm$; let $G^\pm$ denote its $Q$-matrix. (NB. The chains $X, X^+, X^-$ all have finite lifetimes. If you wish to give them decent burial, adjoin a coffin state $\partial$ and be sure to extend functions $f$ on $E$ by making $f(\partial) = 0$.)

The Wiener–Hopf factorization (1.8) holds without any symmetrizability assumption (see Barlow *et al.* 1980). That the whole of the theorem now follows can in principle be gleaned from London *et al.* (1982$b$), but (*mea culpa!*) the treatment there is confusing and misses Parts 1 and 3 of our theorem, and in particular the $*$-adjoints there should be relative to the measure $|V|m$. Here is a quick proof of our present symmetric case.

The matrix $V^{-1}Q$ is diagonalizable with real eigenvalues. One obvious reason is that $V^{-1}Q = L^{-1}(-LV^{-1}L)L$ where $L := (-Q)^{\frac{1}{2}}$, so that $V^{-1}Q$ is similar to the $\langle \cdot, \cdot \rangle$ self-adjoint matrix $(-LV^{-1}L)$. (But of more significance to us is that $V^{-1}Q$ is symmetric relative to the signed inner product $\langle f, Vg \rangle$.) It is a trivial consequence of the fact that each $\delta_i < 0$ that 0 is not an eigenvalue of $V^{-1}Q$.

Suppose now that $V^{-1}Qf = \lambda f$ for some $\lambda < 0$. Then

$$\exp(-\lambda \varphi_t) f(X_t) \quad \text{is a local martingale bounded on} \quad [0, \tau_t^+]. \tag{2.3}$$

The optional-stopping theorem applied at times $\tau_0^+$ and $\tau_t^+$ now shows that (with $f^{\pm}$ denoting the restriction of $f$ to $E^{\pm}$, as usual)

$$\Pi^+ f^+ = f^- \quad \text{and} \quad \exp(tG^+) f^+ = e^{\lambda t} f^+, \quad \text{so that} \quad G^+ f^+ = \lambda f^+. \tag{2.4}$$

Let $\mathscr{V}_p$ (respectively, $\mathscr{V}_n$) be the smallest subspace of $\mathscr{V}$ containing all eigenvectors of $V^{-1}Q$ corresponding to negative (respectively, positive) eigenvalues. It is now clear that

$$\dim(\mathscr{V}_p) = |E^+|, \quad \dim(\mathscr{V}_n) = |E^-|, \tag{2.5}$$

and that
$$f \in \mathscr{V}_p \quad \text{implies} \quad \Pi^+ f^+ = f^- \quad \text{and} \quad G^+ f^+ = (V^{-1}Qf)^+, \tag{2.6}$$

and it is easily checked that

$$f \in \mathscr{V}_p, \quad g \in \mathscr{V}_n \quad \text{imply that} \quad \langle f, Vg \rangle = 0 = \mathscr{E}(f, g). \tag{2.7}$$

Write the decomposition $\mathscr{V} = \mathscr{V}_p \oplus \mathscr{V}_n$ as $f = f_p + f_n$, and define

$$f_+ := f_p^+, \quad f_- := f_n^-. \tag{2.8}$$

For $f^+, g^+ \in \mathscr{V}_+$, define $f^- := \Pi^+ f^+$, $g^- := \Pi^+ g^+$, and then set

$$\langle f^+, g^+ \rangle_+ := +\langle f, Vg \rangle, \quad \mathscr{E}_+(f^+, g^+) := \mathscr{E}(f, g), \tag{2.9}$$

noting that $f^+ = f_+$, $f_- = 0$. Make the analogous definitions with pluses and minuses interchanged.

All the rest *is* plain sailing.

### (c) *The differential equation for* $\Pi^+$

The most obvious way of calculating $\Pi^+$ from the probabilistic problem is via the fact that for $i \in E^-$ and $j \in E^+$, we have $\Pi^+(i,j) = F_j(i, 0)$, where $F_j$ is the unique function on $E \times (-\infty, 0)$ such that

$$\left(V \frac{\partial}{\partial \varphi} + Q\right) F_j(i, \varphi) := V_i \frac{\partial}{\partial \varphi} F_j(i, \varphi) + \sum_k q_{ik} F_j(k, \varphi) = 0, \tag{2.10}$$

where $F_j(n, 0) = \delta_{jn}$ $(n \in E^+)$ and $F_j(\cdot, -\infty) = 0$. Solving (2.10) via separation of variables is effectively the same as the methods used in §§2*a*, *b*.

### (d) *Non-spectral methods; the Riccati equation*

It is important to find methods of calculating $\Pi^{\pm}$ and $G^{\pm}$ which do not entail finding the spectral (eigenvalue–eigenvector) breakdown of $V^{-1}Q$ and then reassembling the components. If $V^{-1}Q$ partitions as

$$V^{-1}Q = \begin{pmatrix} A & B \\ -C & -D \end{pmatrix}, \tag{2.11}$$

then (1.7) shows that (in all cases, not merely in the symmetric case which we are now studying) $\Pi^+$ satisfies the Riccati equation

$$C+\Pi^+A+D\Pi^++\Pi^+B\Pi^+=0. \tag{2.12}$$

Moreover (Williams 1982) $\Pi^+$ is the (componentwise) minimal non-negative solution of (2.12). This idea may be used to give an alternative proof of Theorem 1. The various 'Picard' and Newton–Raphson techniques available for solving (2.12) numerically all focus attention on 'two-dimensional' Green's function representations of $\Pi^+$ such as

$$\Pi^+=\int_0^\infty e^{tD}(C+\Pi^+B\Pi^+)\,e^{tA}\,dt=\int_0^\infty e^{t(D+\frac{1}{2}\Pi^+B)}\,C\,e^{t(A+\frac{1}{2}B\Pi^+)}\,dt. \tag{2.13}$$

In very special situations, Rogers & Williams (1984) were able to utilize a differential equation analogue of (2.12) in the continuous state-space case. The current discussion obviously echoes ideas in stochastic control theory (Whittle 1990, and references therein), but it is not easy to make a direct link, and what are still lacking in the present Wiener–Hopf context are extremality criteria.

## 3. A Riemann–Hilbert problem

### (*a*) *The simplest example*

In discussing this example to indicate something of the flavour, we shall not fuss about the functional analysis of precise domains of definition of operators and forms. We consider the situation when

$$E=\mathbb{R},\quad Qf=\tfrac{1}{2}f'',\quad \langle f,g\rangle=\int_{\mathbb{R}} fg\,dx,$$

$$\mathscr{E}(f,g)=-\langle f,Qg\rangle=\frac{1}{2}\int f'g'\,dx,\quad V(x)=\operatorname{sgn}(x).$$

For $\theta\in\mathbb{R}\backslash\{0\}$, the function $f_\theta$, where

$$f_\theta:=\begin{cases}|\theta|^{-1}\cos\theta x+\theta^{-1}\sin\theta x, & \text{if}\quad x\geqslant 0,\\ |\theta|^{-1}\,e^{|\theta|x}, & \text{if}\quad x\leqslant 0,\end{cases}$$

is a *bounded* solution of $V^{-1}Qf_\theta=-\frac{1}{2}\theta^2f_\theta$. We would like to calculate $\Pi^+$ (in this example, a *stochastic* kernel) from the fact that $\Pi^+f_\theta^+=f_\theta^-$ for $\theta\in\mathbb{R}\backslash\{0\}$.

Let $$\mathbb{H}^+:=\{z\in\mathbb{C}:\mathfrak{J}(z)>0\},\quad \mathbb{H}:=\{z\in\mathbb{C}:\mathfrak{J}(z)\geqslant 0\}. \tag{3.1}$$

Now, for $x<0$ and $z$ in $\mathbb{H}$, define

$$\varphi_x(z)=\int_{[0,\infty)} e^{izy}\,\Pi^+(x,dy), \tag{3.2}$$

so that $\varphi_x$ is analytic on $\mathbb{H}^+$ and continuous on $\mathbb{H}$. We rephrase the $\Pi^+f_\theta^+=f_\theta^-$ property as the 'Riemann–Hilbert problem'

$$\mathfrak{R}[\{|\theta|^{-1}-i\theta^{-1}\}\varphi_x(\theta)]=|\theta|^{-1}\,e^{|\theta|x},\quad \theta\in\mathbb{R}\backslash\{0\}. \tag{3.3}$$

We shall see how to solve this to obtain a result of N. Baker (unpublished):

$$\Pi^+(x,dy)=\Pi(x,y)\,dy,\quad\text{where}\quad \Pi(x,y)=(2|x|y)^{\frac{1}{2}}/\pi(x^2+y^2).$$





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This formula for $\Pi(x,y)$ was also obtained by Rogers & Williams (1984) by showing that the Riccati equation (2.12) takes the form of the Poisson equation

$$\tfrac{1}{2}\Delta\Pi(x,y) = -(4\pi\sqrt{2})^{-1}(|x|y)^{-\frac{3}{2}} \quad (x<0, y>0)$$

in the present context, $\Pi$ being the minimal non-negative solution.

We want $G^+f_\theta^+ = -\tfrac{1}{2}\theta^2 f_\theta^+$ for $\theta \in \mathbb{R}\backslash\{0\}$, and this tells us that $G^+f^+ = \tfrac{1}{2}(f^+)''$ but also that functions $f^+$ in the domain of $G^+$ must satisfy a condition

$$\int_{(0,\infty)} y^{-\frac{3}{2}}[f^+(y)-f^+(0)]\,\mathrm{d}y = 0.$$

The operator $G^+$ must be self-adjoint relative to the $\langle\cdot,\cdot\rangle_+$ product, and we have

$$\langle f^+, g^+\rangle_+ = \int_0^\infty f^+g^+\,\mathrm{d}x - \int_0^\infty\int_0^\infty f^+(x)\,A^+(x,y)\,g^+(y)\,\mathrm{d}x\,\mathrm{d}y,$$

where $$A^+(x,y) = 2\pi^{-2}(xy)^{\frac{1}{2}}(\ln y - \ln x)/(y^2-x^2) \quad (x>0, y>0).$$

### (b) *A more general problem*

For pedagogic purposes (as the French would say), we shall – to avoid all technical difficulties – impose additional assumptions on the situation studied in London *et al.* (1982*a*). By so doing, we destroy the perfect tie-up with Krein's spectral and inverse spectral theory of strings discovered there and developed further in a fine paper by Rogers (1983). (For Krein's theory, see Dym & McKean (1976).) Our purpose here is the different one of seeing how the Riemann–Hilbert aspects of our problem relate to the factorization of $V^{-1}Q$ and of finding a reasonably explicit formula for $\Pi^+$. To achieve this, we need only make a slight modification of London *et al.* (1982*a*).

Here is the situation which we are going to study. We take

$$E = \mathbb{R} \text{ or } E = [a,\infty), \quad \text{where} \quad -\infty < a < 0.$$

For $x$ in $E$, we shall take $V(x) = \operatorname{sgn}(x)$, the value $V(0)$ being irrelevant. Because we shall concentrate on $\Pi^+$ and $G^+$, it is notationally convenient to place 0 in $E^+$ and not in $E^-$, so we take

$$E^+ = [0,\infty), \quad E^- = E \cap (-\infty, 0).$$

We suppose given a continuous strictly positive function $\rho$ on $E^-$ such that for $\delta \in E^-$, $\int_\delta^0 \rho(x)\,\mathrm{d}x < \infty$. We extend $\rho$ to $E$ by taking

$$\rho = 1 \quad \text{on} \quad (0,\infty).$$

The assumption that $\rho = 1$ on $(0,\infty)$ is necessary to make our complex analysis work.

For the purposes of this paper, we take

$$\langle f, g\rangle = \int_E fg\rho\,\mathrm{d}x, \quad \mathscr{E}(f,g) = \int f'g'\,\mathrm{d}x,$$

so that we want $$(Qf)(x) = \tfrac{1}{2}\rho(x)^{-1}f''(x), \quad x \in E\backslash\{0\}.$$

A function $f$ in the domain of $Q$ must have continuous derivative at 0, and in the case when $E = [a,\infty)$, we insist that $f$ also satisfies a non-trivial boundary condition $c_1f'(a) - c_2f(a) = 0$ where $c_1, c_2 \geqslant 0$. (I am not going to become involved in the corresponding precise domain of $\mathscr{E}$. As far as that goes, I have not been absolutely precise about the domain of $Q$ either. Functional analysts must be careful for a number of reasons: the functions $f_\theta$ will never be in $L^2(\rho\,dx)$, for example.)

For $\theta \in \mathbb{R}\backslash\{0\}$, we can find a unique bounded function $f_\theta$ on $E$ such that $f$ is in the domain of $Q$, with $f'(-\infty) = 0$ if $E = \mathbb{R}$, and such that

$$Qf_\theta = -\tfrac{1}{2}\theta^2 V f_\theta, \quad f_\theta'(0) = 1.$$

(Probability theory shows that boundedness *is* the proper restraint on $f$.) The function $f_\theta$ is positive and increasing on $E^-$. We shall of course have

$$f_\theta(y) = f_\theta(0)\cos\theta y + \theta^{-1}\sin\theta y, \quad y \in E^+.$$

If $\varphi_x(x \in E^-)$ is again defined via (3.2) (but for our current $\Pi(x,\cdot)$, of course), we find that in analogy with (3.3),

$$\Re[\{f_\theta(0) - i\theta^{-1}\}\varphi_x(\theta)] = f_\theta(x), \quad \theta \in \mathbb{R}\backslash\{0\}. \tag{3.4}$$

A Riemann–Hilbert problem is essentially one in which one is given a domain $D$ in the complex plane, with boundary $B$, and three functions $a(\cdot), b(\cdot), c(\cdot)$ on $B$ and one wishes to find all functions $\Psi$ analytic in $D$ with continuous extension to $B$ such that

$$\Re[\{a(\theta) + ib(\theta)\}\,\Psi(\theta)] = c(\theta) \quad \text{on } B. \tag{3.5}$$

Now $B$ may have isolated 'bad' points at which some of the functions $a(\cdot), b(\cdot), c(\cdot)$ are discontinuous, and $\Psi$ is only required to have continuous extension to the remaining 'good' points of $B$, and (3.5) is only required to hold at good points. For our problem, $D = \mathbb{H}$ and $B = \mathbb{R} \cup \{\infty\}$, and the boundary point $\infty$ of $B$ is a bad point. Since the triple $(a(\cdot), b(\cdot), c(\cdot))$ can be multiplied by a continuous never-zero function, it may or may not be the case that 0 is a bad point for our problem. (The *index* for our problem is always zero.)

### (c) *A result from complex analysis*

Instead of following explicitly the classic treatment of Riemann–Hilbert problems given by Mushkhelishvili (1946), we shall, as in London *et al.* (1982*a*) let the structure of our problem lead us through. In particular, the problem of calculating $G^+$ forces us to consider the solution $h$ of the homogeneous Riemann–Hilbert problem corresponding to (3.4), and the structure of $h$ is closely connected with the spectral structure of $G^-$. The function $h$ dominates everything.

Adopting this approach means that we shall need only the following facts from complex analysis. Suppose that $u$ is a real-valued harmonic function on $\mathbb{H}^+$ with continuous extension to a set $G$ which contains all but countably many points of the boundary $\mathbb{R}$ of $\mathbb{H}^+$. Suppose further that for some constant $A$ in $(0,\infty)$, we have

$$|u(z)| \leqslant A(1+|z|) \quad \text{for} \quad z \in \mathbb{H},$$

and that, moreover,

$$\sup\{|u(\theta)| : \theta \in G\} < \infty. \tag{3.6}$$

Then, for $z = \alpha + i\beta$ in $\mathbb{H}^+$,

$$u(z) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\beta u(\theta)\,d\theta}{(\theta-\alpha)^2+\beta^2} + b\beta \tag{3.7}$$

for some $b$ in $\mathbb{R}$ and

$$v(z) := \frac{1}{\pi}\int_{\mathbb{R}} \frac{(1+\theta\alpha)(\alpha-\theta)+\theta^2\beta^2}{(\theta-\alpha)^2+\beta^2}\,\frac{u(\theta)\,d\theta}{1+\theta^2} + b\alpha \tag{3.8}$$

defines the unique conjugate function (modulo an additive constant) such that $u+iv$ is analytic on $\mathbb{H}^+$. Standard properties of the Poisson integral as in (say) Lemma 2.11.2 of Dym & McKean (1976) mean that (3.7) follows from the following lemma.

**Lemma.** *Let $u$ be a real-valued harmonic function on $\mathbb{H}^+$ such that for all but countably many $\theta$ in $\mathbb{R}$,*

$$\lim_{\mathbb{H}^+ \ni z \to \theta} u(z) = 0,$$

*and that for some $A$ in $(0, \infty)$,*

$$|u(z)| \leqslant A(1+|z|) \quad \textit{for} \quad z \in \mathbb{H}^+.$$

*Then there exists a constant $c$ in $\mathbb{R}$ such that $u(z) = c\beta$ whenever $z = \alpha + \mathrm{i}\beta \in \mathbb{H}^+$.*

This result is well known, though rarely mentioned in the standard literature. But see Theorem 6.5.4 of Boas (1954). Here is a simple proof which will ease your worries about $z=0$ (which is potentially a bad point in our context).

*Proof.* For $z = \alpha + \mathrm{i}\beta$ in $\mathbb{H}^+$ with $|z| < R$ and for $0 \leqslant \varphi \leqslant \pi$, define

$$P(R, \varphi, z) := \frac{2\beta R(R^2 - |z|^2)\sin\varphi}{\pi |(R\,\mathrm{e}^{\mathrm{i}\varphi} - \alpha)^2 + \beta^2|^2}.$$

The Poisson-integral formula for the semicircle

$$[-R, R] \cup \{R\,\mathrm{e}^{\mathrm{i}\varphi} : 0 \leqslant \varphi \leqslant \pi\}$$

implies that

$$u(z) = \int_0^{\pi} P(R, \varphi, z)\, u(R\,\mathrm{e}^{\mathrm{i}\varphi})\, \mathrm{d}\varphi.$$

Now fix $z_1 = \alpha_1 + \mathrm{i}\beta_1$ and $z_2 = \alpha_2 + \mathrm{i}\beta_2$ in $\mathbb{H}^+$. Then, as $R \to \infty$,

$$\beta_1^{-1} P(R, \varphi, z_1) - \beta_2^{-1} P(R, \varphi, z_2) = O(R^{-3})$$

uniformly over $\varphi$. Hence

$$\beta_1^{-1} u(z_1) - \beta_2^{-1} u(z_2) = O(R^{-1}),$$

and the result is proved. □

It is a simple exercise to modify the argument to establish (3.7) and (3.8) when (3.6) is replaced by the 'natural' condition

$$\int_{\mathbb{R}} (1+\theta^2)^{-1} |u(\theta)|\, \mathrm{d}\theta < \infty.$$

### (*d*) *The function h*

The structure of $\Pi^+$ is inextricably linked to that of the operator $G^+$ and we hope to calculate $G^+$ from the fact that $G^+ f_\theta^+ = -\frac{1}{2}\theta^2 f_\theta^+$ on $(0, \infty)$. It is plausible that for a function $f$ on $[0, \infty)$, $G^+ f = \frac{1}{2}(f)''$, and this is obvious from the probability theory. I have to quote from probability theory that the domain of $G^+$ is specified by two non-negative numbers $p_1, p_2$ and a measure $p_4$ on $(0, \infty)$ via Feller's condition

$$p_1 f(0) - p_2 f'(0) - \int_{(0,\infty)} \{f(y) - f(0)\}\, p_4(\mathrm{d}y) = 0.$$

We hope to determine the triple $(p_1, p_2, p_4)$ modulo scalar multiples via the fact that each $f_\theta^+$ is in the domain of $G^+$.

For $z$ in $\mathbb{H}^+$, let $h(z)$ measure the extent to which the function $y \mapsto e^{izy}$ fails to satisfy Feller's condition, so that

$$h(z) := p_1 - ip_2 z + \int_{(0,\infty)} (1 - e^{izy})\, p_4(dy). \tag{3.9}$$

Then $h$ determines $(p_1, p_2, p_4)$, and $h$ is analytic in $\mathbb{H}^+$ and continuous on $\mathbb{H}$. Moreover, $\Re(h) > 0$ on $\mathbb{H}^+$ and $\Re(h) \geqslant 0$ on $\mathbb{H}$. Thus, in $\mathbb{H}^+$, $\arg(h)$ takes values in $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$. The fact that $f_\theta^+$ satisfies Feller's condition gives us the homogeneous Riemann–Hilbert problem

$$\Re\{[f_\theta(0) - i\theta^{-1}]\, h(\theta)\} = 0, \quad \theta \in \mathbb{R}\backslash\{0\} \tag{3.10}$$

or

$$\arg h(\theta) = -\arctan[\theta f_\theta(0)]$$

except perhaps at 0 and at (at most countably) many points at which $h = 0$. The bounded function $\arg(h)$ is now determined in $\mathbb{H}^+$ by the Poisson-integral formula (3.7) with $b = 0$ and the conjugate function $\ln|h|$ is determined on $\mathbb{H}^+$ modulo an additive constant. Hence $h$ is determined on $\mathbb{H}$ modulo a multiplicative constant. This argument, essentially due to J. F. C. Kingman (unpublished), was given in London *et al.* (1982*a*).

Note that $h$ is real on the imaginary axis and that for a variety of reasons,

$$h(\alpha + i\beta) = \overline{h(-\alpha + i\beta)}, \quad \alpha + i\beta \in \mathbb{H}. \tag{3.11}$$

## (*e*) *Total monotonicity of* $p_4$

I now quote one of the central results from London *et al.* (1982*a*), namely that the measure $p_4$ for our problem has a totally monotone density relative to Lebesgue measure: for some measure $J$ on $(0, \infty)$,

$$p_4(dy) = dy \int_{(0,\infty)} e^{-ry}\, J(dr). \tag{3.12}$$

For our current special problem, this fact derives from the self-adjointness of $Q$ relative to $\langle \cdot, \cdot \rangle$ in a manner reminiscent of Reuter (1956) and Kingman (1967). See also Rogers (1983) for a complex-analytic explanation combining classical fluctuation theory with the theory of Pick functions.

The algebra in London *et al.* (1982*b*) shows how the total monotonicity of $p_4$ is profoundly related to the self-adjointness of $G^-$ relative to $\langle \cdot, \cdot \rangle_-$, the measure $J$ arising from the spectral projection-valued measure for $G^-$. (However, the fact that $\langle \cdot, \cdot \rangle_-$ is not an $L^2(m)$ inner product makes the fact that $J$ is a positive measure require extra structure present in our problem.)

## (*f*) *Some simple estimates*

We need the following facts:

$$\text{the measure } J \text{ is non-zero}; \tag{3.13}$$

for $x \in E^-$,

$$0 < f_\theta(x) \leqslant f_\theta(0)\, \text{sech}\,(\tfrac{1}{2}\theta x A_x), \tag{3.14}$$

where $A_x := [\inf\{\rho(w) : x \leqslant w < \frac{1}{2}x\}]^{\frac{1}{2}} > 0$;

$$\text{the even function } \theta \mapsto f_\theta(0) \text{ is decreasing in } |\theta|. \tag{3.15}$$





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Proofs of (3.13) and (3.15) are left to you.

*Proof of* (3.14). Let $x \in E^-$, let $I_x$ denote the interval $[x, \frac{1}{2}x]$, and, for $u \in I_x$, define

$$g_\theta(u) := \cosh \theta A_x(u-x).$$

Then, on $I_x$, both $g_\theta$ and $f_\theta$ are positive, and

$$(g_\theta f'_\theta - f_\theta g'_\theta)' = \theta^2 f_\theta g_\theta(\rho - A_x^2) > 0.$$

Since, also,
$$(g_\theta f'_\theta - f_\theta g'_\theta)(x) = f'_\theta(x) \geqslant 0,$$
we see that $f_\theta / g_\theta$ is non-decreasing on $I_x$, whence

$$f_\theta(x) = \frac{f_\theta(x)}{g_\theta(x)} \leqslant \frac{f_\theta(\frac{1}{2}x)}{g_\theta(\frac{1}{2}x)} \leqslant \frac{f_\theta(0)}{\cosh(\frac{1}{2}x\theta A_x)}. \qquad \square$$

*Remark.* Result (3.14) is probabilistically obvious.

From (3.9) and (3.12), we have

$$h(z) = p_1 - \mathrm{i}p_2 z - \int_{(0,\infty)} \frac{\mathrm{i}zK(\mathrm{d}r)}{r - \mathrm{i}z}, \quad \text{where} \quad K(\mathrm{d}r) := r^{-1}J(\mathrm{d}r). \tag{3.16}$$

If $z = \alpha + \mathrm{i}\beta \in \mathbb{H}$, we therefore have

$$\Re\{h(z)\} = p_1 + p_2\beta + \int \frac{\beta(r+\beta)+\alpha^2}{(r+\beta)^2+\alpha^2} K(\mathrm{d}r) \geqslant p_1 + p_2\beta + \frac{1}{2}\int \frac{\alpha^2+\beta^2}{r^2+\alpha^2+\beta^2} K(\mathrm{d}r), \tag{3.17}$$

so that $\inf(\Re\{h(z)\} : z \in \mathbb{H}^+;\ |z| \geqslant 1) > 0$.

If $p_1 \neq 0$, then $|z^{-1}h(z)| \to \infty$ as $|z| \to 0$, while if $p_1 = 0$, then, for $0 < |z| \leqslant 1$, we have

$$\Re\{z^{-1}h(z)\} = \int \frac{(\beta+r)K(\mathrm{d}r)}{(r+\beta)^2+\alpha^2} \geqslant \frac{1}{2}\int \frac{rK(\mathrm{d}r)}{r^2+1} > 0, \tag{3.18}$$

whence $\inf(\Re\{z^{-1}h(z)\} : z \in \mathbb{H};\ 0 < |z| \leqslant 1) > 0$.

### (*g*) *Solving for* $\Pi^+$

Fix $x$ in $E^-$. Recall the definition (see (3.2)) of $\varphi_x$ on $\mathbb{H}$, and note that $|\varphi_x| \leqslant 1$ on $\mathbb{H}$. Define

$$\Psi_x(z) = -\mathrm{i}zh(z)^{-1}\varphi_x(z), \quad z \in \mathbb{H}\backslash\{0\}, \tag{3.19}$$

pausing to note that for real $\mu > 0$,

$$\Psi_x(\mathrm{i}\mu) = \left(\frac{p_1}{\mu} + p_2 + \int \frac{K(\mathrm{d}r)}{r+\mu}\right)^{-1} \int_{[0,\infty)} \mathrm{e}^{-\mu y}\Pi^+(x, \mathrm{d}y). \tag{3.20}$$

Since $|\varphi_x| \leqslant 1$ on $\mathbb{H}$, the estimates obtained in subsection (*f*) show that $\Psi_x$ which is analytic in $\mathbb{H}^+$ and continuous on $\mathbb{H}\backslash\{0\}$ satisfies

$$|\Psi_x(z)| \leqslant A(1+|z|) \quad (z \in \mathbb{H}) \tag{3.21}$$

for some constant $A$ in $(0, \infty)$. However, it is immediate from (3.4) and (3.10) that

$$\Re\Psi_x(\theta) = f_\theta(x)R(\theta) \quad (\theta \in \mathbb{R}\backslash\{0\}),$$

where
$$R(\theta) := \theta^2\Re\{h(\theta)^{-1}\} \quad (\theta \in \mathbb{R}\backslash\{0\}).$$

We note that $R(\theta)$ and $f_\theta(\xi)$ are even in $\theta$; recall (3.11). It is obvious that $R(\theta) = O(\theta^2)$, while from (3.14), $f_\theta(x)$ tails off exponentially in $\theta$. Hence, by the Poisson-integral formula, we have, for some $b$ in $\mathbb{R}$,

$$\Psi_x(\mathrm{i}\mu) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\mu}{\mu^2+\theta^2} f_\theta(x) R(\theta)\,\mathrm{d}\theta + b\mu. \tag{3.22}$$

Because of the probabilistically obvious fact (proved analytically at Lemma 24 of London *et al.* (1982*a*)) that $\Pi^+(x,\cdot)$ attaches no mass to the singleton $\{0\}$,

$$\int \mathrm{e}^{-\mu y}\Pi^+(x,\mathrm{d}y) \to 0 \quad \text{as} \quad \mu\to\infty,$$

and it is now clear that from (3.20) and (3.22) that $b = 0$. It is further clear from (3.20) and (3.22) that $\Pi^+(x,\mathrm{d}y)$ has a smooth density $\Pi(x,y)$ relative to Lebesgue measure on $[0,\infty)$ which is the convolution of two functions $F$ and $H_x$ on $[0,\infty)$, where

$$F(y) := p_1 + p_2\,\delta(y) + \int_{(0,\infty)} \mathrm{e}^{-ry}K(\mathrm{d}r) = p_1 + p_2\,\delta(y) + p_4(y,\infty) \quad (y>0), \tag{3.23}$$

$$H_x(y) := 2\pi^{-1}\Re\left\{\int_{(0,\infty)} \mathrm{e}^{\mathrm{i}\theta y} f_\theta(x) R(\theta)\,\mathrm{d}\theta\right\} \quad (y>0). \tag{3.24}$$

Note that while $F$ takes only positive values, $H_x$ may take both positive and negative values.

The lesson is that the convolution description discovered by McGregor (1988, 1990) for Example 4*b* below holds generally. The somewhat puzzling relation between this convolution description and other convolution descriptions of $\Pi^+$ known from probability is something I hope to discuss in a sequel.

### (*h*) *Symmetry*

In analogy with (1.5), we have

$$\Pi^-(y,\mathrm{d}x) = \Pi(x,y)\,\rho(x)\,\mathrm{d}x \quad \text{for} \quad y\in E^+, \quad x\in E^-. \tag{3.25}$$

This is no longer so obvious!

## 4. Examples, methods, and their interrelations

### (*a*) *The canonical example*

Let us make a minor generalization of the simplest example in §3*a* by taking

$$E = \mathbb{R}, \quad \rho = 1 \text{ on } E^+ = [0,\infty), \quad \rho = c^2 \text{ on } E^- = (-\infty,0), \quad \text{where} \quad c\in(0,\infty).$$

Then $f_\theta(x) = (c|\theta|)^{-1}\exp(c|\theta|x)$ for $x<0$. We find from (3.10) that

$$\arg\{h(\theta)\} = -\operatorname{sgn}(\theta)\,\tfrac{1}{2}\pi\alpha, \quad \text{where} \quad 0<\alpha<1, \quad \tan\tfrac{1}{2}\pi\alpha = c^{-1}. \tag{4.1}$$

Hence, we may take

$$h(z) = (-\mathrm{i}z)^\alpha, \quad \text{so that} \quad p_1 = p_2 = 0 \quad \text{and} \quad F(y) = p_4(y,\infty) = y^{-\alpha}/\Gamma(1-\alpha). \tag{4.2}$$

Then $R(\theta) = |\theta|^{-\alpha}\cos\frac{1}{2}\pi\alpha$, so that, by an easily justified calculation,

$$H_x(y) = 2\pi^{-1}c^{-1}\Gamma(2-\alpha)\,\Re\{(c|x|+\mathrm{i}y)^{\alpha-2}\}\cos\tfrac{1}{2}\pi\alpha.$$

Since
$$\frac{\mathrm{d}}{\mathrm{d}u}\left\{\frac{1}{\beta\gamma}\left(\frac{u}{\delta u+\beta}\right)^{\gamma}\right\} = \frac{1}{(\delta u+\beta)^{1+\gamma}\,u^{1-\gamma}},$$
we have
$$\int_0^y u^{-\alpha}\{c|x|+\mathrm{i}(y-u)\}^{\alpha-2}\,\mathrm{d}u = \frac{y^{1-\alpha}(c|x|)^{\alpha-1}}{(1-\alpha)\,(c|x|+\mathrm{i}y)}.$$

Hence the convolution formula for $\Pi^+$ works out easily to yield Baker's formula (Rogers & Williams 1984; McGill 1989*a*):

$$\Pi^+(x,\mathrm{d}y) = \Pi(x,y)\,\mathrm{d}y, \quad \text{where} \quad \Pi(x,y) = \frac{2c^{\alpha}|x|^{\alpha}y^{1-\alpha}\sin\frac{1}{2}\pi\alpha}{\pi(c^2x^2+y^2)}. \tag{4.3}$$

McGill's reason for calling this the canonical example is explained below. For the simplest example, we have $c = 1$ and $\alpha = \frac{1}{2}$.

### (*b*) *An example with reflecting boundary*

We now take
$$E = [-1,\infty), \quad \rho = 1 \text{ on } E^+ = [0,\infty), \quad \rho = c^2 \text{ on } E^- = [-1,0),$$
and impose the boundary condition $f'(-1) = 0$ on functions in the domain of $Q$. Then
$$f_\theta(x) = \{c\theta\sinh c\theta\}^{-1}\cosh c\theta(x+1), \quad x\in E^-,$$
$$\arg\{h(\theta)\} = -\arctan(c^{-1}\coth c\theta) \quad (\theta\in\mathbb{R}).$$
We may take
$$h(z) = -\mathrm{i}z\,\Gamma(\tfrac{1}{2}\alpha-\mathrm{i}zc\pi^{-1})\,\pi^{\alpha-1}/\Gamma(1-\tfrac{1}{2}\alpha-\mathrm{i}zc\pi^{-1}),$$
whence $p_1 = p_2 = 0$ and $F(y) = (\frac{1}{2}c^{-1}\pi)^{\alpha}/\Gamma(1-\alpha)\,(\sinh\frac{1}{2}\pi c^{-1}y)^{\alpha}$, where $\tan\frac{1}{2}\pi\alpha = c^{-1}$, $\alpha\in(0,1)$. Note that $\arg h(\theta) \sim -\operatorname{sgn}(\theta)\frac{1}{2}\pi\alpha$ as $|\theta|\to\infty$, and that
$$h(z) \sim (-\mathrm{i}z)^{\alpha} \quad \text{as} \quad |z|\to\infty \quad \text{in } \mathbb{H},$$
$$F(y) \sim y^{-\alpha}/\Gamma(1-\alpha) \quad \text{as} \quad y\downarrow 0.$$

(Compare (4.1) and (4.2).) One can carry out the convolution calculation in this case (McGregor 1988, 1990) to obtain

$$\Pi(x,y) = \frac{c^{-1}(\sin\frac{1}{2}\pi|x|)^{\alpha}\,(\sinh\frac{1}{2}\pi c^{-1}y)^{1-\alpha}\cosh(\frac{1}{2}\pi c^{-1}y)}{\sin^2\frac{1}{2}\pi x+\sinh^2\frac{1}{2}\pi c^{-1}y}\sin\tfrac{1}{2}\pi\alpha. \tag{4.4}$$

This result for $\Pi$ was first obtained for the case $c = 1$ in London *et al.* (1982*b*) and for general $c$ by Baker (1984).

### (*c*) *Conformal mapping; elliptic functions*

The formula (4.4) was guessed by Neil Baker (then my research student) via a heuristic inferred from comparing examples (*a*) and (*b*) in the (then known) case when $c = 1$. He used contour integration (Baker 1984) to show that $\Pi(x,y)$ as at (4.4) does satisfy the equation $\Pi^+f_\theta^+ = f_\theta^-$ $(\theta\in\mathbb{R}\backslash\{0\})$. That the solution of this equation is unique was known from London *et al.* (1982*a*).

McGill (1989*a*) makes precise an application of conformal mapping which works (and justifies Baker's heuristic) in a limited set of cases which, in the way of mathematics, contains several cases of great importance. McGill is thereby able to use conformal mapping to derive (4.4) rigorously from what he calls the canonical case (4.3). The conformal mapping effects changes of variable which map the calculations in example (*a*) into those done by McGregor in example (*b*).

McGill's really striking use of the 'Baker heuristic' is to solve the case when $E = [a, b]$, where $a < 0 < b$ and we impose reflecting boundary conditions at $a$ and $b$ for functions in the domain of $Q$, where $\rho = 1$ on $(0, b)$ and $\rho = c^2$ on $[-a, 0)$, and where, finally, $V$ is the sgn function. As had been conjectured by Baker, the solution is a combination of elliptic functions similar to the combination of hyperbolic and trigonometric functions at (4.4), these latter functions being limiting cases of the doubly periodic elliptic functions. For a different viewpoint on the heuristic, see also (McGill 1989*b*).

### (*d*) *The Riccati equation*

Suppose now that

$$E = \mathbb{R} \quad \text{or } [a, \infty) \text{ or } [a, b] \quad (a < 0 < b),$$

that $\rho$ is any strictly positive continuous function on $E$ and that $V = \text{sgn}$ on $E$. We no longer require that $\rho = 1$ on $(0, \infty)$. Rogers & Williams (1984) showed that under these circumstances, for $x \in E^-$, $y \in E^+$, we have

$$\Pi^+(x, \mathrm{d}y) = \Pi(x, y)\,\rho(y)\,\mathrm{d}y, \quad \Pi^-(y, \mathrm{d}x) = \rho(x)\,\Pi(x, y)\,\mathrm{d}x,$$

where the function $\Pi$ on $E^- \times E^+$ satisfies the Riccati equation:

$$(Q_x + Q_y)\,\Pi(x, y) = -\text{const.}\,p(y)\,q(x) \tag{4.5}$$

for some non-negative functions $p$ on $E^+$ and $q$ on $E^-$, which are certain linear functionals of $\Pi$. In the cases considered previously, $p$ is the density relative to Lebesgue measure of the measure $p_4$.

If, for example, $E = \mathbb{R}$, $\beta > -1$ and $c > 0$ and

$$\rho(x) = \begin{cases} x^\beta, & \text{if} \quad x > 0, \\ -c^{2+\beta}|x|^\beta, & \text{if} \quad x < 0, \end{cases}$$

then scaling properties may be combined with (4.5) to yield

$$\Pi(x, y) = K|x|^\alpha y^{1-\alpha}/(c^{2+\beta}|x|^{2+\beta} + y^{2+\beta}) \quad (x \in E^-, y \in E^+),$$

where $\alpha$ is the unique solution in $(0, 1)$ of the equation

$$\sin((1-\alpha)\,\delta) = c \sin(\alpha\delta), \quad \text{where} \quad \delta := \pi/(2+\beta),$$

and

$$K = \pi^{-1}(2+\beta)\,c^\alpha \sin(\alpha\delta)$$

(see Rogers & Williams 1984). As McGill (1989*a*) states, and as Rogers and I were aware, one could also solve this case via classical Wiener–Hopf factorization of an asymmetric stable process. Note that when $\beta = 0$, we are back with example (*a*). McKean (1963) had solved the case when $\beta = 1$ and $c = 1$.

Though equation (4.5) can only be used to calculate $\Pi(x, y)$ in the presence of something akin to scaling properties, it clearly throws a lot of light on when solutions may be obtained via Baker–McGill change of variables, and it is interesting to look afresh at McGill (1989*a*). Of course, equation (4.5) has the potential for multi-dimensional generalization which other methods do not.

### (*e*) *An example with absorbing boundary*

We now take (with subscripts 'abs' used to highlight the absorbing case)

$$E_{\text{abs}} = [-1, \infty), \quad \rho_{\text{abs}} = 1 \text{ on } E^+, \quad \rho_{\text{abs}} = c^2 \text{ on } E^-$$





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where $c \in (0, \infty)$, and impose the absorbing boundary condition $f(-1) = 0$ on functions $f$ in the domain of $Q_{\text{abs}}$. Then, with $\alpha := 2\pi^{-1} \arctan(c^{-1})$, as usual,

$$\Pi_{\text{abs}}(x, y) = \frac{c^{-1}(\sin \frac{1}{2}\pi|x|)^{\alpha} (\sinh \frac{1}{2}\pi c^{-1} y)^{1-\alpha} \cos \frac{1}{2}\pi x}{\sin^2 \frac{1}{2}\pi x + \sinh^2 \frac{1}{2}\pi c^{-1} y} \sin \frac{1}{2}\pi\alpha \tag{4.6}$$

(see Baker 1984; McGill 1989*a*).

### (*f*) *A three-dimensional example*

Let

$$E = \mathbb{R}^3, \quad E^+ = \{w \in E : |w| \geqslant 1\}, \quad E^- = \{w \in E : |w| < 1\}.$$

Let $c \in (0, \infty)$, and define

$$\rho = c^2 \text{ on } E^-, \quad \rho = 1 \text{ on } E^+, \quad V = -1 \text{ on } E^-, \quad V = 1 \text{ on } E^+.$$

For nice functions $f$ and $g$ on $E$, we take

$$\langle f, g \rangle = \int_E fg\rho \, \mathrm{d}w, \quad \mathscr{E}(f, g) = \frac{1}{2} \int_E (\operatorname{grad} f) \cdot (\operatorname{grad} g) \, \mathrm{d}w,$$

so that $Q = \frac{1}{2}\rho^{-1}\Delta$.

I now explain how to begin the study of this case, concentrating only on the 'radial part' of the problem. The information we obtain about the radial part can be utilized in the study of the angular part; but that is a story for another occasion.

The 'radial' definitions are obvious. We define

$$E_{\text{rad}} = [0, \infty), \quad E^+_{\text{rad}} = [1, \infty), \quad E^-_{\text{rad}} = [0, 1),$$

and, for functions $f$ and $g$ on $[0, \infty)$, we define (omitting $4\pi$!)

$$\langle f, g \rangle_{\text{rad}} = \int_{[0,\infty)} fg\rho_{\text{rad}} \, \mathrm{d}r,$$

$$\mathscr{E}_{\text{rad}}(f, g) = \frac{1}{2} \int_{[0,\infty)} f'g'r^2 \, \mathrm{d}r = \frac{1}{2} \int_{[0,\infty)} (rf)'(rg)' \, \mathrm{d}r, \tag{4.7}$$

where $f' := \mathrm{d}f/\mathrm{d}r$, etc., and

$$\rho_{\text{rad}} = r^2 \lambda(r), \quad \text{where} \quad \lambda = 1 \text{ on } E^+_{\text{rad}}, \quad \lambda = c^2 \text{ on } E^-_{\text{rad}}.$$

The reason for the final expression at (4.7) is that for 'test functions' $f$ and $g$,

$$r^2 f'g' = (rf)'(rg)' - (rfg)'.$$

We see that we need to take

$$Q_{\text{rad}}\, g = \tfrac{1}{2}\lambda(r)^{-1} r^{-1} (rg)''.$$

To be sure, this is a convoluted way to arrive at the radial part of the laplacian!

For a function $g$ on $[0, \infty)$, define $\varphi g$ on $[-1, \infty)$ by

$$(\varphi g)(u) := (u+1)\, g(u+1).$$

Then with $Q_{\text{abs}}$ as in §4*e*,

$$Q_{\text{rad}} = \varphi^{-1} Q_{\text{abs}} \varphi; \tag{4.8}$$

and, formally at least, this 'similarity', a special case of the Doob $h$-transform, shows that for $x \in E^-_{\text{rad}}$ and $y \in E^+_{\text{rad}}$,

$$\Pi^+_{\text{rad}}(x, \mathrm{d}y) = \{x^{-1}\Pi_{\text{abs}}(x-1, y-1)\, y^{-1}\}\, y^2\, \mathrm{d}y, \tag{4.9}$$

where $\Pi_{\text{abs}}$ is as at (4.6). I skip for now rigorous proof of (4.9). As a first step towards convincing yourself, you should check that

$$\int_{[1,\infty)} \Pi^+_{\text{rad}}(x, \mathrm{d}y) = 1, \quad x \in [0, 1).$$

Of course, if $f^+$ is a radial function on $E^+$ for the three-dimensional problem, then if $f^+(v) = R^+(|v|)$ for $v$ in $E^+$, we have for $w$ in $E^-$,

$$(\Pi^+ f^+)(w) = \int_{[1,\infty)} \Pi^+_{\text{rad}}(x, \mathrm{d}y)\, R^+(y), \quad x = |w|.$$

## 5. Concluding remarks

I have tried to convey something of the flavour of the non-probabilistic aspects of part of probabilistic Wiener–Hopf Theory. Because I have written the paper for non-probabilists, I have not discussed the relation between the problems considered here and classical fluctuation theory. (Bingham 1975; Greenwood & Pitman 1980; Rogers 1983, 1984; McGill 1989*a*) make excellent reading for this, and include better complex analysis.

The complex analysis in *this* paper has been adequate for the theory but good fun only in the examples; and do read McGill (1989*a*, 1990). While complex analysis is exactly right for fluctuation theory and for many other applications of Wiener–Hopf theory (Noble 1959), its use for our problem, while great fun, seems to me contrived in that it applies only to 'cooked' problems (some of which are, however, very important). The heart of the matter is equation (1.8).

The title is designed to allow a sequel 'Some further aspects of ...', which may be more appropriate to a specialist-field journal. The 'Wiener–Hopf theory with noise' which Joanne Kennedy and I (1990) have begun to study might well prove interesting, and will involve further complex analysis and hard explicit calculations. Of course it is the case that doing explicit calculations is more in the spirit of 19th- than 20th-century mathematics. But such explicit answers as have been obtained have been invaluable for making, and for shooting down, conjectures.

My debt to co-authors of various papers is self-evident. I thank Joanne Kennedy for her help. I began this work at Swansea, and it is a pleasure to acknowledge the continuing help I receive from there.

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